Performance of Joint Equalization and Trellis-Coded Modulation on Multipath Fading Channels

Mao-Ching Chiu and Chi-chao Chao

Abstract— **In the literature the performance of joint maximum-likelihood sequence estimation for trellis-coded modulation systems was analyzed under the assumption that fading is so slow that the channel does not change during all error events. In this paper we extend the performance analysis to general fading processes by considering the correlation function of the time-variant channel impulse response instead of assuming constant fading. An easily evaluated closed-form upper bound is derived for the pairwise error probability. The bit error rate is then estimated by using a truncated union bound. Computer simulations show that our analytical results are good for all cases considered especially when diversity reception is used.**

I. Introduction

The maximum-likelihood sequence estimation (MLSE) originally proposed in [1] is an effective equalization technique to combat intersymbol interference (ISI). The performance of MLSE was first analyzed in [1] [2] for uncoded systems on linear time-invariant channels. Since exact expressions on error probability are difficult to obtain, upper and lower bounds were employed for performance evaluation.

Channel coding has been known for a long time as an effective way to combat the effect of noise. Trellis-coded modulation (TCM) [3] can achieve a significant coding gain on additive white Gaussian noise channels without sacrificing the channel bandwidth. The optimum decoding for TCM on ISI channels is the joint MLSE which combines equalization and decoding in the maximum-likelihood sense.

For frequency-nonselective fading channels, the system performance for trellis-coded MPSK was derived in [4] under the assumption of ideal interleaving or constant fading gains. Performance analysis of trellis-coded PSK or DPSK on correlated fading channels can be found in [5]–[7].

On frequency-selective fading channels with diversity reception, the performance of joint MLSE for TCM systems was analyzed in [8][9] under the assumption that fading is so slow that the channel remains unchanged during all error events. However, this condition is difficult to hold for long error events, and the time-variant characteristics of the fading channel are ignored completely in the analysis with this assumption. In this paper we extend the performance analysis to general fading processes by, instead of assuming constant fading, considering the correlation function of the time-variant channel impulse response. The channel is assumed to be slowly varying so that the channel estimator can completely track the variation. An easily evaluated closed-form upper bound is derived for the pairwise error probability. The bit error rate is then estimated by using a truncated union bound. Analytical and simulation results are presented for land-mobile channels with different fading rates. We find that our analytical results are good for all cases considered especially when diversity reception is used.

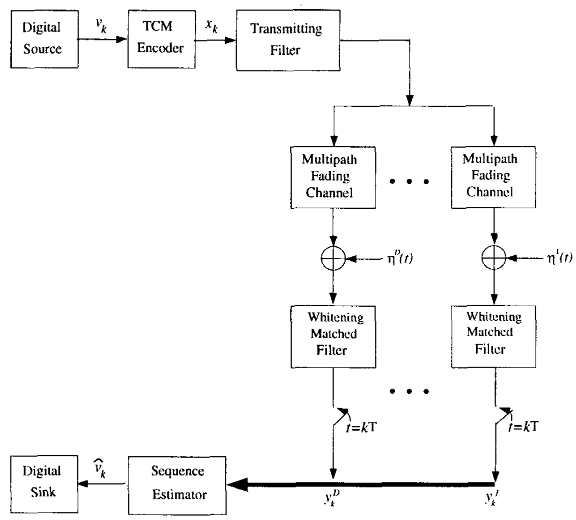

Fig. 1. Model of a data transmission system employing TCM over a multipath fading channel.

II. System and Channel Models

Our system model is shown in Fig. 1. The whole channel is modeled as D independent fading channels each corrupted by additive white Gaussian noise. Assume during each signaling interval T, n information bits are inputed to the system, denoted by v_k. The output signal x_k of the TCM encoder first goes through the transmitting filter for waveform shaping and is then transmitted over the multipath fading channel. At the receiver, a whitening matched filter is used to process the received signal to enhance the signal-to-noise ratio at the sampling instant $t = kT$.

Paper approved by Evangelos Eleftheriou, the Editor for Equalization and Coding of the IEEE Communications Society. Manuscript received February 23, 1994; revised October 14, 1994 and January 3, 1995. This work was supported by the National Science Council of Republic of China under Grant No. NSC 81-0404-E-007-015. This work was presented in part at the 1992 National Symposium on Telecommunications, Chungli, Taiwan, R.O.C., December 1992 and in part at the 1993 IEEE International Symposium on Information Theory, San Antonio, Texas, January 1993.

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IEEE Log Number 9411574. 0090-6778/95$4.00

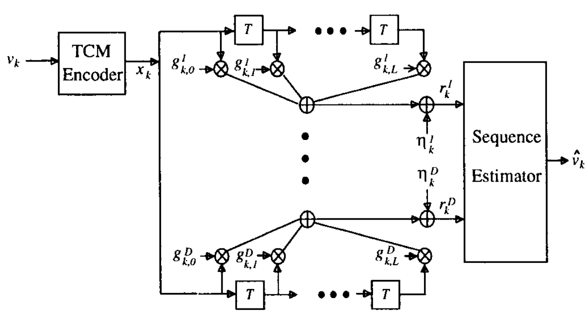


Fig. 2. Equivalent discrete-time model.

The equivalent discrete-time model of the whole system is shown in Fig. 2, where $g_{k,i}^d$, $0 \le i \le L$, $1 \le d \le D$, denotes the ith tap coefficient of the dth diversity branch at time k. The received signal can be expressed as

$$r_k^d = \sum_{i=0}^{L} x_{k-i} g_{k,i}^d + \eta_k^d,$$

where L is the length of channel memory which depends on the channel delay spread and $\{\eta_k^d\}$ are i.i.d. zero-mean complex Gaussian random variables with variance $\sigma_{\eta_k^d}^2 = (1/2)\mathrm{E}\{|\eta_k^d|^2\} = N_0/2$. Assume that all the diversity branches are wide-sense stationary with Rayleigh fading. Hence $\{g_{k,i}^d\}$ are modeled as zero-mean complex Gaussian random variables. Since D diversity branches are assumed to be independent, $\{g_{k,i}^d\}$ are independent in index d. But they may be correlated in time index k and delay index i. The correlations depend not only on the fading channel characteristics but also on the transmitter and receiver filters.

The equivalent discrete-time model can be modeled as a finite-state machine. The combined state is the concatenation of the encoder state and the channel state. Let φ_k be the encoder state at time k. Then the state of the whole finite-state machine can be defined as

$$s_k = (x_{k-L}, x_{k-L-1}, \ldots, x_{k-1}; \varphi_k),$$

or equivalently, in terms of the information sequence as

$$s_k = (\varphi_{k-L}; v_{k-L}, v_{k-L+1}, \ldots, v_{k-1}). \tag{1}$$

The TCM encoder and the discrete-time channel model hence lead to a combined ISI and code trellis. From (1), since there are 2^{nL} ISI states, for a S-state TCM, the total number of states in the combined trellis is $S2^{nL}$. The sequence estimator uses the Viterbi algorithm to find the maximum-likelihood information sequence over the combined ISI and code trellis.

III. Performance Analysis

Let $\mathbf{v} = \{v_k\}$ be the transmitted information sequence and $\mathbf{v} \oplus \mathbf{e} = \{v_k \oplus e_k\}$ be the estimate of the information sequence at the receiver output. Since $\{x_k\}$ is the transmitted signal sequence of the information sequence $\{v_k\}$, let $\{\tilde{x}_k\} = \{x_k - \varepsilon_k\}$ denote the corresponding signal sequence of $\{v_k \oplus e_k\}$. By employing the union-bounding technique, the average bit error rate for an MLSE receiver can be bounded by

$$P_b \le \frac{1}{n} \sum_{\mathbf{e} \in \mathbf{E}} W_b(\mathbf{e}) \sum_{\mathbf{v}} \mathrm{P}\{\mathbf{v}\} \mathrm{P}\{\Gamma(\mathbf{v} \oplus \mathbf{e}) \ge \Gamma(\mathbf{v}) | \mathbf{v}\}, \tag{2}$$

where $\mathbf{E}$ is the set of all error events, $W_b(\mathbf{e})$ is the number of bit errors of the error sequence $\mathbf{e}$, $\mathrm{P}\{\mathbf{v}\}$ is the a prior probability of the transmitted information sequence $\mathbf{v}$, and $\Gamma(\mathbf{v})$ is the path metric of $\mathbf{v}$.

Assume the channel estimates are perfect as in [8][9], i.e., the estimates of the tap coefficients equal the actual values, which means that the channel considered is changing slowly enough such that the channel estimator can completely track the variation. For an error event of length l, the conditional pairwise error probability is given by

$$\begin{aligned}
&\mathrm{P}\{\Gamma(\mathbf{v} \oplus \mathbf{e}) \ge \Gamma(\mathbf{v}) | \mathbf{v}, \mathbf{g}\} \\
&= \mathrm{P}\left\{ \sum_{d=1}^{D} \sum_{k=1}^{l} \left| r_k^d - \sum_{i=0}^{L} \tilde{x}_{k-i} g_{k,i}^d \right|^2 \right. \\
&\qquad \left. \le \sum_{d=1}^{D} \sum_{k=1}^{l} \left| r_k^d - \sum_{i=0}^{L} x_{k-i} g_{k,i}^d \right|^2 \right\} \\
&= \frac{1}{2} \mathrm{erfc}\left(\sqrt{\frac{\sum_{d=1}^{D} \sum_{k=1}^{l} |a_k^d|^2}{4N_0}} \right),
\end{aligned} \tag{3}$$

where

$$a_k^d = \sum_{i=0}^{L} \varepsilon_{k-i} g_{k,i}^d. \tag{4}$$

The pairwise error probability is the average of (3) over the fading characteristics. This average can be written as the probability that a quadratic form of correlated complex Gaussian random variables is less than a fixed number, which can then be computed exactly by using the methods in [5][7][10][11, Append. B][12]. Nevertheless, to obtain an easily evaluated closed-form expression, we use the upper bound $(1/2)\mathrm{erfc}(x) \le (1/2)\exp(-x^2)$ instead. Then the pairwise error probability can be bounded by

$$\mathrm{P}\{\Gamma(\mathbf{v} \oplus \mathbf{e}) \ge \Gamma(\mathbf{v}) | \mathbf{v}\} \le \frac{1}{2} \prod_{d=1}^{D} \Omega_d, \tag{5}$$

where

$$\Omega_d = \int \exp\left(-\frac{\sum_{k=1}^{l} |a_k^d|^2}{4N_0} \right) p(\mathbf{a}^d) d\mathbf{a}^d,$$

and $\mathbf{a}^d = [a_1^d, a_2^d, \ldots, a_l^d]^T$. Assume that all diversity branches have the same fading statistics, and hence Ω_d are the same for all d. So in the following derivation, we ignore the subscript d completely for simplicity. Define

$$\Omega = \int \exp\left(-\frac{\sum_{k=1}^{l} |a_k|^2}{4N_0} \right) p(\mathbf{a}) d\mathbf{a}. \tag{6}$$

Let $\mathbf{g}_i = [g_{1,i}, g_{2,i}, \ldots, g_{l,i}]^T$ denote the collection of the ith tap coefficients from time 1 to l. And let the correlation matrix $\mathbf{R}_{i,j}$ be given by $\mathbf{R}_{i,j} = (1/2)\mathrm{E}\{\mathbf{g}_i\mathbf{g}_j^H\}$. Then from (4), the random vector $\mathbf{a} = [a_1, a_2, \ldots, a_l]^T$ is jointly Gaussian with covariance matrix

$$\mathbf{\Lambda}' = \frac{1}{2}\mathrm{E}\left\{\mathbf{a}\mathbf{a}^H\right\} = \sum_{i=0}^{L}\sum_{j=0}^{L}\mathbf{A}_i\mathbf{R}_{i,j}\mathbf{A}_j^H,$$

where

$$\mathbf{A}_i = \begin{bmatrix} \varepsilon_{1-i} & 0 & \cdots & 0 \\ 0 & \varepsilon_{2-i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_{l-i} \end{bmatrix}.$$

Since in general the matrix $\mathbf{\Lambda}'$ may be singular, i.e., $a_1, a_2, \ldots, a_l$ may be "degenerate" in the sense of [13, p. 87]. We show in Appendix that if the channel considered is truly randomly time-variant, the only case that $\mathbf{a}$ is degenerate is when some components of $\mathbf{a}$ are always equal to zero. Without loss of generality, assume that after relabeling those zero elements of $\mathbf{a}$ are $a_{m+1}, \ldots, a_l$. Let $\mathbf{\Lambda}$ be the resulting $m \times m$ submatrix of $\mathbf{\Lambda}'$ by deleting all the zero rows and columns which correspond to $a_{m+1}, \ldots, a_l$. Thus $\mathbf{\Lambda}$ is the nonsingular covariance matrix of nondegenerate elements in $\mathbf{a}$. Let $\mathbf{z} = [a_1, a_2, \ldots, a_m]^T$ and $f = \sum_{k=1}^{m}|a_k|^2 = \mathbf{z}^H\mathbf{z}$. Then f is a Hermitian quadratic form of complex Gaussian random variables. As shown in [11, Append. B], the characteristic function of f is given by

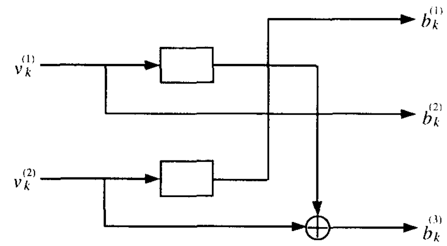
$$G_f(s) = \frac{1}{\det(2s\mathbf{\Lambda}^* + \mathbf{I})},$$

where $\mathbf{\Lambda}^*$ denotes the complex conjugate of $\mathbf{\Lambda}$. We hence obtain

$$\begin{aligned} \Omega &= G_f(s)|_{s=1/(4N_0)} \\ &= \frac{1}{\det\left(\frac{1}{2N_0}\mathbf{\Lambda}^* + \mathbf{I}\right)} = \frac{1}{\det\left(\frac{1}{2N_0}\mathbf{\Lambda} + \mathbf{I}\right)}, \end{aligned} \tag{7}$$

where the third equality follows since $\mathbf{\Lambda}$ is Hermitian. This can be substituted back to (5) to obtain an upper bound for the pairwise error probability.

It is clear from (2) that an infinite series needs to be computed to obtain a true upper bound on bit error rate. Since the transfer-function bounding technique is not applicable here for multipath fading channels, in computation of the series only a finite number of terms can be included, which results in a truncated union bound. Evaluation of the truncated upper bound can be done by using the error-state diagram and the stack algorithm which are similar to those in [8][9]. Though rigorously this can only be considered as an approximation to the true upper bound, simulation results obtained show that our analytical results based on the truncated bound are good for all cases considered. Actually we have shown in [14] the local convergence (i.e., for sufficiently high signal-to-noise ratios (SNR)) of our true upper bound for the special case that $\{g_{k,i}^d\}$ are independent in index k and index i. Though we are unable to give a general proof, we do expect that our upper bound is locally convergent for the cases considered in the paper.



$b_k^{(1)}$	$b_k^{(2)}$	$b_k^{(3)}$	x_k
0	0	0	0
0	0	1	1
0	1	0	2
0	1	1	3
1	0	0	4
1	0	1	5
1	1	0	6
1	1	1	7

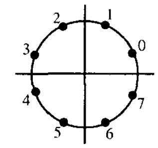


Fig. 3. Convolutional encoder, signal constellation, and signal labeling for a 4-state trellis-coded 8-PSK scheme.

IV. Analytical and Simulation Results

In the following an example in a two-tap ($L = 1$) channel is given. The TCM scheme considered is shown in Fig. 3. For land mobile channels, the correlation function of channel tap coefficients can be modeled as a Bessel function of order zero [15]:

$$\frac{1}{2}\mathrm{E}\left\{g_{k,i}^* g_{k+l,j}\right\} = \begin{cases} 0, & i \neq j, \\ \sigma_i^2 J_0(2\pi f_D l T), & i = j, \end{cases}$$

where f_D is the Doppler spread which depends on the velocity of the mobile and σ_i^2 is the variance of the ith tap coefficient described by the multipath intensity profile. Note that in this example we assume that the coefficients in different taps are uncorrelated; while in general they may not be. We set $\sigma_0^2 = 1$, $\sigma_1^2 = 1$, and the normalized Doppler spread $f_D T = 0.05$, 0.02, 0.005 or 0.0005 for each diversity branch. The tap coefficients in simulation are generated by passing white Gaussian sequences through proper shaping filters as in [16, Append. A]. The average received bit signal-to-noise ratio is given by $E_b/N_0 = \sum_{i=0}^{L}\mathrm{E}\left\{|g_{k,i}|^2\right\}\left(\mathrm{E}\left\{|x_k|^2\right\}/n\right)/N_0$.

In this example, the stack algorithm terminates when N terms in the series (2) are included in the sum such that the following two conditions are satisfied: a) N is larger than a preset threshold number and b) the last $N/2$ terms

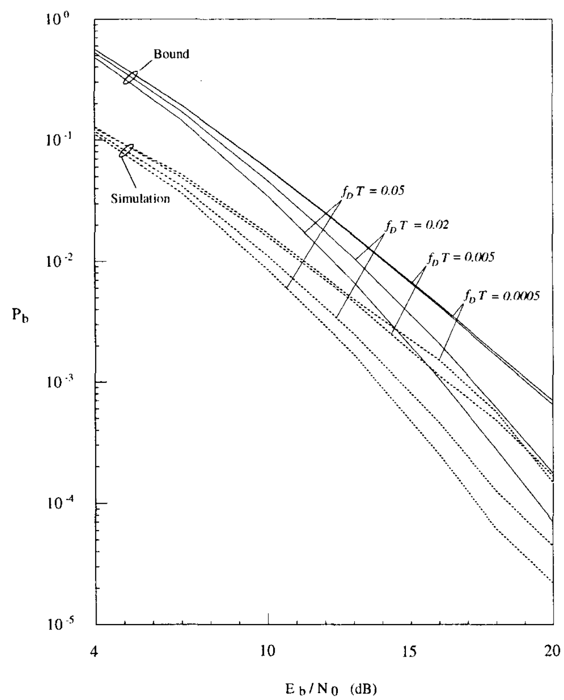


Fig. 4. Analytical and simulation results for a 16-state 8-PSK MLSE on multipath Rayleigh fading channels with $f_D T = 0.05$, 0.02, 0.005, 0.0005 and $D = 1$.

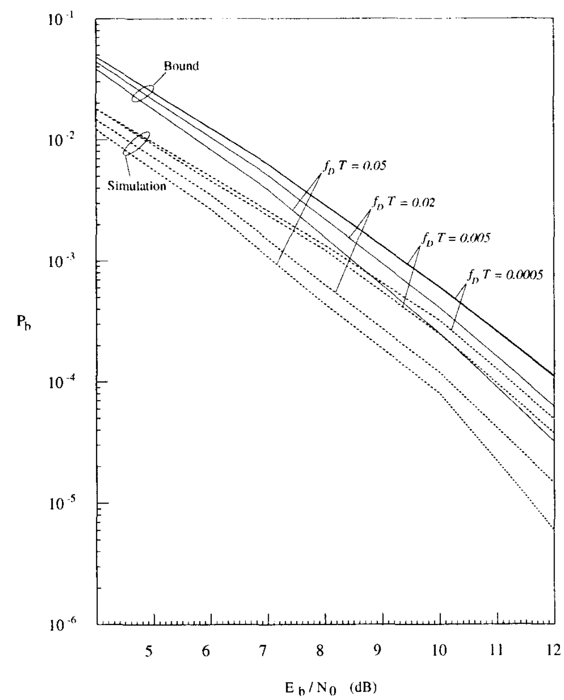


Fig. 5. Analytical and simulation results for a 16-state 8-PSK MLSE on multipath Rayleigh fading channels with $f_D T = 0.05$, 0.02, 0.005, 0.0005 and $D = 2$.

contribute less than 0.1% of the sum of the first $N/2$ terms. For all numerical results obtained, more than 80,000 terms are included in the sum. Analytical and simulation results for this system are given in Fig. 4 and Fig. 5 for $D = 1$ and $D = 2$, respectively. We find that our analytical results are within 1 to 2.5 dB of the simulation results at high SNR. Furthermore, we can see that the performance deteriorates as the normalized Doppler spread decreases and consequently the correlation between time-adjacent tap coefficients increases. However, interpretation of this should be made carefully since the results obtained are based on the assumption that fading is slow enough such that the channel estimator can completely track the variation. Intuitively, as fading becomes faster, this might not be true because of the limited tracking capability of the channel estimator. Nevertheless, the results obtained are applicable over the range of slow fading where the channel estimator can work well.

V. Concluding Remarks

In this paper an easily evaluated closed-form upper bound for the pairwise error probability is derived for joint MLSE in TCM systems with diversity reception on multipath Rayleigh fading channels. The bit error rate is then estimated by using a truncated union bound. Although we use an upper bound on pairwise error probability, computer simulations show that our analytical results are tight for systems considered especially when diversity reception is available. We observe that the correlation function of the channel impulse response plays an important part and affects the bit error rate performance of the system.

In our analysis, if we let $L = 0$ and $D = 1$, the channel will be reduced to one with frequency-nonselective fading and only one diversity branch. Hence the results in [4][6] about Rayleigh fading with perfect channel state information and with or without interleaving become special cases of our result. Also for this case our numerical results will be close to those derived in [7] where the exact expression for the pairwise error probability is used. If we set the normalized Doppler spread $f_D T = 0$, our channel model will be reduced to that considered in [8][9].

In this paper we assume that perfect channel estimates are available. For fast fading channels, this might not be practical because of the limited tracking capability of the channel estimator. The performance of adaptive MLSE with imperfect channel estimates over multipath fading channels will be studied in a subsequent paper.

Appendix
Condition for Degeneration of $\mathbf{a}$

Consider the following elements of $\mathbf{a}$, which are rewritten here for convenience.

$$a_j = \sum_{i=0}^{L} g_{j,i}\varepsilon_{j-i}, \quad \text{for } j = 1, \ldots, l.$$

If the error sequence $\varepsilon_1, \ldots, \varepsilon_l$ contains more than L consecutive zeros, then some random variables in $\mathbf{a}$ will always be equal to zero. This never happens for uncoded systems; however, it may sometimes happen for TCM systems. Assume that the discrete model considered is truly randomly time-variant such that random variables in $\mathbf{g} = (g_{1,0}, \ldots, g_{1,L},\ g_{2,0}, \ldots, g_{2,L}, \ldots, g_{l,0}, \ldots, g_{l,L})$ are nondegenerate, i.e., each random variable in $\mathbf{g}$ can not be expressed as a linear combination of others. If such a linear combination does exist, then it will hold at all times; hence this condition is valid for most random models considered. In the following we will show that the only case for degeneration of $\mathbf{a}$ is that some components in $\mathbf{a}$ are always equal to zero, or equivalently, there are more than L consecutive zeros in the error sequence. Suppose, say, a_k is a dependent random variable and is not always equal to zero. Then a_k can be written as a linear combination of nondegenerate random variables a_{s_j}, for $j = 1, \ldots, m$:

$$a_k = \sum_{j=1}^{m} \gamma_j a_{s_j},$$

where γ_j, $j = 1, \ldots, m$, are not all zeros. Since a_k does not degenerate to zero, there exists $\varepsilon_{k-i} \neq 0$ for some $0 \leq i \leq L$. Assume without loss of generality that $\varepsilon_k \neq 0$; then

$$g_{k,0} = \frac{1}{\varepsilon_k}\left(\sum_{j=1}^{m}\sum_{i=0}^{L} g_{s_j,i}\varepsilon_{s_j-i}\gamma_j - \sum_{i=1}^{L} g_{k,i}\varepsilon_{k-i}\right),$$

which implies that $g_{k,0}$ can be expressed as a linear combination of others in $\mathbf{g}$, a contradiction.

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